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In the space between real numbers

by Richard O'Donovan [*]

Real numbers

Sometimes I wonder: are real numbers really what they seem to be? ℝ is complete: any set bounded above has a least upper bound. But that is not the end of the story. Apart from being complete, the real numbers have the property - shared with rational numbers - of being dense i.e., between any two distinct real (rational) numbers, there is a real (rational) number distinct from the two first.

Now consider a geometric line as per Euclid (and as undefined objects as in Euclid). Establish two points on this line and define this to be a unit. Now scatter rational numbers on this line in an orderly manner, with 0 on the fixed point to the left and 1 on the fixed point to the right. All this done in a very usual way for setting the rational line.

Do these points completely cover the geometric line?

A consequence of density is that in the rationals, there is no immediate successor. A consequence of this consequence is that a rational, set on the geometric line, cannot touch another rational, say on its right. Otherwise, the latter would be the immediate successor of the former. Yet, however far we zoom in by an integer factor, this space is impossible to see. But we know this is true because there are irrational numbers in the space between rational numbers. Rational numbers cannot measure this space since it would be the measure of an interval between two consecutive numbers - which do not exist. But consider $\sqrt{2}$: there is space to fit it in the rationals, hence there was space, not necessarily a big space, but a space. If we zoom in, as long as we can see two distinct rational numbers, we can see infinitely many rational numbers. But there is space, so it must be possible to zoom in in such a way that we can see only one rational number. But then if we imagine that $\sqrt{2}$ fits in the space, if we can see $\sqrt{2}$ and another rational number, then we can see infinitely many numbers. Hence, because $\sqrt{2}$ is not a rational number, there is empty space, but because of density, if we visualise this space, we cannot see any other rational or irrational number.

Standing on the line

Imagine you are a point on a rational line and you look ahead. If you could see a point, it would hide all others. That would be the next point... Hence from one point, no other point can be seen. But this is not surprising once one notices that by being on a point implies to have dimension zero, so any nonzero distance will seem beyond the horizon. If you are a point on a line composed of rational numbers, you cannot know in which direction the line goes.

It is not because we cannot give a size to this space that this space does not exist and indeed, rational numbers are not fine enough to measure this space (neither are real numbers). And of course, this space is not between two consecutive rational numbers, since such a situation cannot occur. It is a space around the rational numbers which cannot be written as the size of an interval of numbers of any sort. When we write that this space is "between" rational numbers, we simply intend to indicate that it is in their midst. And the same arguments hold if "rational" is replaced by "real": there can be no next real number hence real numbers cannot cover the line.

The axiom of choice

If one wishes to deny the axiom of choice, one is free to do so. But then, one consequence is that infinite cardinals cannot be compared. It is not possible to state that the real numbers have a larger cardinality than the rational numbers. In the construction due to Robinson and Luxemburg one assumes the mainstream view with the axiom of choice. Then it is possible to extend filters to ultrafilters and thus construct the hyperreal system in which there are indeed numbers (hyperreals) in the space between real numbers. In turn, these hyperreals do not cover the line, but they live among real numbers without touching anything! In the world where rationals are countable and reals are uncountable, there are hyperreals infinitely close to real numbers.

α-theory

Pick a point (any point) on the line in the space around zero on the positive side. Call it α and define its distance from zero to be "infinitely small". Define addition and multiplication in a way similar to ideals, and with a few adjustments, this leads to α-theory, which is a very straightforward method of providing infinitely small numbers and due to Benci, Di Nasso and Forti.

Extending the axioms

Since infinitesimals exist at least in the hyperreal mode, attempts to provide the same advantages but on an axiomatic basis emerged, as given by Nelson or Hrbacek. In these approaches, an axiom states that there are nonstandard objects and a consequence is that there are nonstandard real numbers. A curious thing happens here. The internal view is that the extra axioms form a syntactic extension: no new object is formed but it becomes possible to state new properties about already existing objects. The "exists" in "there exists nonstandard numbers" is interpreted as meaning "among the real numbers already defined, some can be said to be nonstandard." In this view, nothing extra is fitted in the space around real numbers. The external view, on the other hand, admits that "there exists" is interpreted as "there are extra objects" and the standard real numbers are the usual real numbers, the nonstandard fit in the free space. The difference is, in fact, purely philosophical and the resulting mathematics are the same in both interpretations. Yet, it may seem strange to realise that the internal view does not contradict that real numbers are all there is whereas the external view does. Here, we adopt the external view in which the usual real numbers are now the standard numbers and there are ultrasmall numbers.

Other approaches

There are other ways of defining numbers which fit in the space. Either way, it would be a waste of space to leave all the line to standard real numbers.

Walking the line and Zeno's paradox revisited

Now imagine a vehicle rolling on the rational line. If time is continuous (whatever this means), travelling means passing all points one after the other. But this is impossible. Hence the question: how can one travel along a rational line? How can one travel along a real line? Back to Zeno: how can one travel at all? [1] For every instant in time, there is a corresponding position on the line, but this does not explain how one moves. Because mathematicians at the time of set theoretic formalisation were concerned with completeness, they thought they had everything and dubbed the real numbers as the continuum which is terrible on philosophical and mathematical grounds. It is the continuum by definition and not by proof - which would have required a definition of the continuum not referring to real numbers.

To say that a movement is determined by a relation between position given by a real coordinate and an instant given by a real value is simply stating a relation between real numbers and real numbers. This does not analyse what movement is but merely how movement is currently measured. Standing on the point of the line given by a rational (real) number, what happens if one starts to move? First rolling in the empty space, then eventually a next point is met. So movement along a rational (real) line is impossible. Yet we accept that movement in a straight line does exist...

In our view, the major difference between a line in the real world and a line in the mathematical world, is that in the real world, the physical one in which we live, read, walk and do mathematics, a line is not infinitely divisible. If density means that it is possible to divide by two any number of times, then the distance between you and me is not a dense set. When we reach the quantum scale, things jump from one state to the next. And here is the crucial phrase: "from one state to the next"! Movement is possible because there are quanta of space (and also of time) hence it is possible to go from one position to the next, from one instant to the next. An answer to Zeno is that, unlike numbers, the real world is not infinitely divisible. Quanta are a necessity for movement. But Zeno can be excused: he did not know about quanta.

Ultrasmall numbers and infinitesimals as representatives of quanta

In the hyperreal system, chose N to be a hypernatural number. Then $1/N = dx$ which is infinitely small, written $dx \simeq 0$. In the approach developed by Hrbacek, Lessmann and the author; relative to any number there is an ultrasmall real number i.e., any number can be taken as standard "ground level". Even if we look at an ultrasmall interval, we are simply in a smaller theatre and every concept of density and proximity happens again. Let N be an ultralarge natural number (hence nonstandard) and $dx = 1/N$ which is ultrasmall, written $dx \simeq 0$. Whichever theory is favoured by the reader, let $1/N = dx \simeq 0$.

Let $a_0 = 0$; $a_k = a_0 + k \cdot dx$, then $\{a_i | 0 \leq i \leq N\}$ is a set which partitions the interval [0; 1] into N equal ultrasmall intervals of length $dx \simeq 0$. Each interval $[a_k\ ;\ a_{k+1}]$ contains infinitely many rational and irrational numbers but it is possible to hop from a_k to a_{k+1} as if dx were a quantum, since even though we have $a_k \simeq a_{k+1}$, the second point is the next in the partition. Zoom in so that you can see dx as the length of your step. The line in front of you is dense with nonstandard real numbers. Take a step of size dx, skip over infinitely many nonstandard points and reach the next partition point. And so the line can be travelled by hopping quanta.

Nonstandard analysis provides a framework where movement can be explained, Zeno's paradox solved and quanta have a mathematical correspondence - even if it is more abstract (in the sense that it does not correspond to a measure) than an $\hbar$.

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[1] It is sometimes said that modern analysis with the concept of limit answers Zeno's paradox. This is a gross oversight that the limit is not necessarily attained, so the arrow that goes half way, then half of half way, etc, does have a limit which is the bullseye, but the sequence does not reach its limit, just as $f: x \mapsto 1/x$ does not reach its asymptotes. (Zeno understood this perfectly)

References: This paper is a reflection on mathematics based on research but not specific to any paper in particular. References were made to A. Robinson, W.A Luxemburg, E. Nelson, K. Hrbacek, M. Di Nasso, V. Benci, M. Forti and Zeno of course.

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Un gioco…

(*L. Corso*) Nel libro "Il Gioco" di Manfred Eigen (premio Nobel per la Chimica nel 1967) e Ruthild Winkler, edizioni Adelphi, Milano, 1986, si afferma che la popolazione mondiale ha avuto a partire dal 1750 una crescita di tipo iperbolico. L'affermazione impressiona e confuta quella diffusa in certi ambienti scientifici secondo cui la popolazione mondiale cresce con un andamento esponenziale smorzato (si veda il noto modello della funzione logistica). Allo scopo di verificare con i dati posseduti oggi quanto espresso dai due ricercatori dell'Istituto Max Planck di Göttingen scelgo (per tentativi) il seguente modello (questo modello non è riportato nel libro citato. I calcoli e il grafico sono stati fatti con Mathematica della Wolfram Research):

$$\hat{y} = a + \frac{b}{x - x_0}, \quad con \quad 0 < x < x_0 . \qquad (1)$$

Determiniamo i parametri a e b con il metodo dei minimi quadrati:

$$S(a, b) = \sum\nolimits_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum\nolimits_{i=1}^{n} \left(y_i - a - \frac{b}{x_i - x_0} \right)^2 ; \qquad (2)$$

dove della funzione $S(a, b)$ ci interessa il minimo, y_i è la popolazione al tempo x_i e (x_i, y_i) sono le coppie di valori sperimentali osservate, $\hat{y}_i$ sono i valori teorici calcolati in base al modello, x_0 è il limite temporale critico per una popolazione che tende all'infinito. I dati riportati nel testo citato con l'aggiunta di quelli più recenti (controllati, approssimati e depurati da grossolane precarietà interpretative) sono:

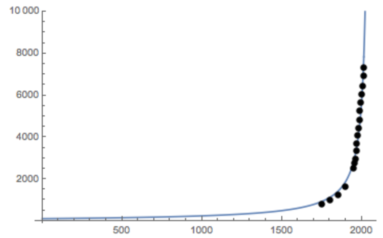
x_i (*anni*)	1750	1800	1850	1900	1950	1955	1960
y_i ($pop \times 10^6$)	791	978	1262	1650	2519	2756	2982
x_i (*anni*)	1965	1970	1975	1980	1985	1990	1995
y_i ($pop \times 10^6$)	3335	3692	4068	4435	4831	5263	5674
x_i (*anni*)	2000	2005	2010	2015.			
y_i ($pop \times 10^6$)	6070	6454	6930	7349.			

Da (2), dopo aver calcolato le derivate parziali di S rispetto ad a e a b e averle poste uguali a zero, si ottiene il seguente sistema di equazioni:

$$\begin{cases} n\,a + \left(\sum\nolimits_{i=1}^{n} \dfrac{1}{x_i - x_0} \right) b = \sum\nolimits_{i=1}^{n} y_i \\ \left(\sum\nolimits_{i=1}^{n} \dfrac{1}{x_i - x_0} \right) a + \left(\sum\nolimits_{i=1}^{n} \left(\dfrac{1}{x_i - x_0} \right)^2 \right) b = \sum\nolimits_{i=1}^{n} \left(\dfrac{y_i}{x_i - x_0} \right) \end{cases}$$

e sostituendo e risolvendo (dando a x_0 il valore virtuale di 2050) si ha: $\hat{y} \approx -31.9083 + 288127/(2050 - x)$. Calcoliamo l'indice di accostamento I_2 che è, in sostanza, un coefficiente di variazione dei valori sperimentali rispetto al modello teorico:

$$I_2 = \frac{\sqrt{\dfrac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{n}}}{\dfrac{\sum_{i=1}^{n} \hat{y}_i}{n}} .$$



Esso, in base ai dati a disposizione, è $I_2 \cong 0.0861$.
Il valore dimostra che il modello iperbolico (1) si accosta abbastanza bene ai dati sperimentali. Perciò, la tesi di Eigen e Wilkler è sostenibile. È naturale che questa tendenza dovrà smorzarsi, essendo le risorse della Terra finite. Si provi, infine, ad accostare sugli stessi dati un modello esponenziale e si verifichi con I_2 se l'adattamento risulta migliore. Nel grafico appena sopra, sull'asse delle ascisse poniamo gli anni e sull'asse delle ordinate la popolazione in milioni di individui.